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High accuracy difference schemes for a class of three space dimensional singular parabolic equations with variable coefficients

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Abstract

In this article, two-level implicit difference methods of $O(k^2 + h^4)$ using 19-spatial grid points for the solution of three space dimensional heat conduction equation and unsteady Navier–Stokes' equations in polar coordinates are proposed. Unconditionally stable ADI methods for the solution of the heat conduction equation in polar coordinates are also discussed. Numerical examples given here show that the methods developed here retain their order and accuracy everywhere including the region in the vicinity of the singularity $r = 0$. © 1997 Elsevier Science B.V. All rights reserved.

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1. Introduction

The numerical solution of the heat conduction equation and unsteady Navier–Stokes' equations in polar coordinates are of great importance in problems of heat transfer and viscous fluid flow. Unsteady Navier–Stokes' equations in polar coordinates often arise in the mathematical modelling used to solve problems in fluid dynamics involving turbulence. The current engineering requirement for computational fluid dynamics codes for realistic viscous flow problems have provided the impetus for the development and implementation of high-order difference methods. It has been repeatedly demonstrated on model problems, that even the simplest types of high-order methods should provide tremendous practical advantages in terms of diminishing the required number of points and also the overall computing time for a desired solution. Ciment et al. [1] have discussed high-order

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operator compact implicit methods for parabolic equations. An $O(k^2 + h^4)$ difference method for three space dimensional heat conduction equation in cylindrical polar coordinates was given by Iyengar and Manohar [3], Marshall and Spiegel [7] have integrated the Navier–Stokes' equations by means of an explicit operator. A second-order numerical method for integrating two-dimensional Navier–Stokes' equations for a viscous incompressible, laminar duct flow which is applicable for low Reynolds number only was discussed by Roscoe [10]. Ellison et al. [2] have presented a new low-order finite difference scheme for solving two space dimensional transient, incompressible unsteady Navier–Stokes' problems on a bounded simply-connected region. Jain et al. [4–6] and Mohanty and Jain [8] derived fourth-order difference methods for the system of two and three space dimensional elliptic and parabolic equations and solved Navier–Stokes' equations in cartesian coordinates. Recently, Mohanty and Jain [9] have developed fourth-order difference methods for the system of three-dimensional elliptic equations with variable coefficients and solved three-dimensional Navier–Stokes' equations in cylindrical polar coordinates, in which the numerical results confirmed the order and accuracy even in the vicinity of the singularity $r = 0$ for high Reynolds number.

For studying viscous fluid flow phenomena, one faces problems of solving three space dimensional unsteady Navier–Stokes' equations in polar coordinates. The numerical solutions of Navier–Stokes' equations in cartesian coordinates were already discussed. But little has been done for handling three space dimensional unsteady Navier–Stokes' equations in polar coordinates. The difficulties have been experienced in the past for the fourth-order numerical solution of three space dimensional unsteady Navier–Stokes' equations in polar coordinates. The solution usually deteriorate in the neighbourhood of the singularity $r = 0$. In this article, we propose a finite-difference method of $O(k^2 + h^4)$ for a class of linear singular initial boundary value problem, where $h>0$ and $k>0$ are grid sizes in space and time directions, respectively. We refine our procedure in such a way that the solutions retain the order and accuracy even in the vicinity of the singularity. A further refinement allows us to obtain unconditionally stable alternating direction implicit (ADI) methods which are of $O(k^2 + h^4)$. Such ADI methods require the solution of tridiagonal systems of equations parallel to coordinate axes, at each time step, independent of the order of the method. A separate difference equation of $O(k^2 + h^4)$ is given for $r = 0$. The same technique is then applied to obtain $O(k^2 + h^4)$ two-level implicit difference method for the solution of three space dimensional unsteady Navier–Stokes' equations in cylindrical polar coordinates. It is mentioned here that no two-level difference methods of $O(k^2 + h^4)$ using 19-spatial grid points are known for solving three space dimensional unsteady Navier–Stokes' equations in polar coordinates. It is shown here that for a fixed $\lambda = k/h^2$, the proposed methods are of $O(h^4)$ in space.

2. Mathematical details of the methods

Let us first consider a procedure to construct high-order difference schemes for the equation

$$u_{xx} + B(x)u_{yy} + C(x,y)u_{zz} = u_t + D(x)u_x + E(x,y)u_y + G(x,y,z,t). \tag{1}$$

As usual, let us assume that the solution domain $\Omega \equiv \{(x,y,z,t) \mid 0<x,y,z<1, t>0\}$ is covered by a set of grid points with spacing $h>0$ and $k>0$ in space and time directions, respectively. The grid points (x, y, z, t) are given by $x = x_l = lh$, $y = y_m = mh$, $z = z_n = nh$, $t = t_j = jk$; $l, m, n = 0(1)N+1$ and $j = 0, 1, \ldots,$ where N is a positive integer. Let the exact solution values of u, G, B, C, D and E at the grid point $(x_l, y_m, z_n, t_j) \equiv (l, m, n, j)$ are denoted by $U^j_{l,m,n}$, $G^j_{l,m,n}$, B_l, $C_{l,m}$, D_l and $E_{l,m}$, respectively. Let $u^j_{l,m,n}$ be the approximate value of $U^j_{l,m,n}$.

The initial and boundary conditions are given by

$$u(x, y, z, 0) = u_0(x, y, z), \tag{2a}$$

$$u(0, y, z, t) = G_0(y, z, t), \qquad u(1, y, z, t) = G_1(y, z, t), \tag{2b}$$

$$u(x, 0, z, t) = H_0(x, z, t), \qquad u(x, 1, z, t) = H_1(x, z, t), \tag{2c}$$

$$u(x, y, 0, t) = I_0(x, y, t), \qquad u(x, y, 1, t) = I_1(x, y, t). \tag{2d}$$

Now, we simply follow the approaches given by Mohanty and Jain [8]. We start with the following approximations:

For $a, b, c = 0, \pm 1$, let

$$\bar{t}_j = t_j + \frac{k}{2}, \tag{3a}$$

$$\bar{u}^j_{l,m,n} = (u^{j+1}_{l,m,n} + u^j_{l,m,n})/2, \tag{3b}$$

$$\bar{u}^j_{tl+a,m+b,n+c} = (u^{j+1}_{l+a,m+b,n+c} - u^j_{l+a,m+b,n+c})/k, \tag{3c}$$

$$\bar{u}^j_{xl,m+b,n+c} = (\bar{u}^j_{l+1,m+b,n+c} - \bar{u}^j_{l-1,m+b,n+c})/(2h), \tag{3d}$$

$$\bar{u}^j_{yl+a,m,n+c} = (\bar{u}^j_{l+a,m+1,n+c} - \bar{u}^j_{l+a,m-1,n+c})/(2h), \tag{3e}$$

$$\bar{u}^j_{xl\pm1,m,n} = (\pm 3\bar{u}^j_{l\pm1,m,n} \mp 4\bar{u}^j_{l,m,n} \pm \bar{u}^j_{l\mp1,m,n})/(2h), \tag{3f}$$

$$\bar{u}^j_{yl,m\pm1,n} = (\pm 3\bar{u}^j_{l,m\pm1,n} \mp 4\bar{u}^j_{l,m,n} \pm \bar{u}^j_{l,m\mp1,n})/(2h), \tag{3g}$$

$$\bar{u}^j_{xxl,m+b,n} = (\bar{u}^j_{l+1,m+b,n} - 2\bar{u}^j_{l,m+b,n} + \bar{u}^j_{l-1,m+b,n})/(h^2), \tag{3h}$$

$$\bar{u}^j_{yyl+a,m,n} = (\bar{u}^j_{l+a,m+1,n} - 2\bar{u}^j_{l+a,m,n} + \bar{u}^j_{l+a,m-1,n})/(h^2), \tag{3i}$$

$$\bar{u}^j_{zzl+a,m+b,n} = (\bar{u}^j_{l+a,m+b,n+1} - 2\bar{u}^j_{l+a,m+b,n} + \bar{u}^j_{l+a,m+b,n-1})/(h^2), \tag{3j}$$

$$\bar{G}^j_{l+a,m+b,n+c} = G(x_{l+a}, y_{m+b}, z_{n+c}, \bar{t}_j). \tag{3k}$$

Let us denote

$$S_{pq} = \frac{\partial^{p+q} S(x_l, y_m)}{(\partial x)^p (\partial y)^q}, \quad S = B, C, D \text{ and } E. \tag{4}$$

Now, using the approximations (3.a)–(3.b), notation (4) and the method developed by Mohanty and Jain [8], we may obtain a difference scheme of $O(k^2+h^4)$ for solving Eq. (1) as

$$\Big[6\delta_x^2+p_1\delta_y^2+p_2\delta_z^2+2p_3\delta_y^2\mu_x\delta_x+2p_4\delta_z^2\mu_x\delta_x+2p_5\delta_z^2\mu_y\delta_y$$

$$+p_6\delta_x^2\delta_y^2+p_7\delta_y^2\delta_z^2+p_8\delta_z^2\delta_x^2\Big]\bar{u}^j_{l,m,n}$$

$$=\frac{h^2}{2}\Big[6(\bar{u}^j_{tl,m,n}+D_l\bar{u}^j_{xl,m,n}+E_{l,m}\bar{u}^j_{yl,m,n}+\bar{G}^j_{l,m,n})$$

$$+(1-hq_1)(\bar{u}^j_{tl+1,m,n}+D_{l+1}\bar{u}^j_{xl+1,m,n}+E_{l+1,m}\bar{u}^j_{yl+1,m,n}+\bar{G}^j_{l+1,m,n})$$

$$+(1+hq_1)(\bar{u}^j_{tl-1,m,n}+D_{l-1}\bar{u}^j_{xl-1,m,n}+E_{l-1,m}\bar{u}^j_{yl-1,m,n}+\bar{G}^j_{l-1,m,n})$$

$$+(1-hq_2)(\bar{u}^j_{tl,m+1,n}+D_l\bar{u}^j_{xl,m+1,n}+E_{l,m+1}\bar{u}^j_{yl,m+1,n}+\bar{G}^j_{l,m+1,n})$$

$$+(1+hq_2)(\bar{u}^j_{tl,m-1,n}+D_l\bar{u}^j_{xl,m-1,n}+E_{l,m-1}\bar{u}^j_{yl,m-1,n}+\bar{G}^j_{l,m-1,n})$$

$$+\bar{u}^j_{tl,m,n+1}+D_l\bar{u}^j_{xl,m,n+1}+E_{l,m}\bar{u}^j_{yl,m,n+1}+\bar{G}^j_{l,m,n+1}$$

$$+\bar{u}^j_{tl,m,n-1}+D_l\bar{u}^j_{xl,m,n-1}+E_{l,m}\bar{u}^j_{yl,m,n-1}+\bar{G}^j_{l,m,n-1}$$

$$+hq_1B_{00}(\bar{u}^j_{yyl+1,m,n}-\bar{u}^j_{yyl-1,m,n})+hq_1C_{00}(\bar{u}^j_{zzl+1,m,n}-\bar{u}^j_{zzl-1,m,n})$$

$$+hq_2(\bar{u}^j_{xxl,m+1,n}-\bar{u}^j_{xxl,m-1,n})+hq_2C_{00}(\bar{u}^j_{zzl,m+1,n}-\bar{u}^j_{zzl,m-1,n})$$

$$+h^2q_3\bar{u}^j_{yyl,m,n}+h^2q_4\bar{u}^j_{zzl,m,n}\Big], \tag{5}$$

where

$$\begin{aligned}
&p_1=6B_{00}+\frac{h^2}{2}B_{20},\quad p_2=6C_{00}+\frac{h^2}{2}(C_{20}+C_{02}),\\
&p_3=hB_{10}/2,\ p_4=hC_{10}/2,\ p_5=hC_{01}/2,\ p_6=(1+B_{00})/2,\\
&p_7=(B_{00}+C_{00})/2,\ p_8=(1+C_{00})/2,\ q_1=D_{00}/2,\\
&q_2=E_{00}/(2B_{00}),\ q_3=D_{00}B_{10},\ q_4=D_{00}C_{10}+2q_2C_{01}
\end{aligned} \tag{6}$$

and

$$\mu_xu_l=(u_{l+\frac{1}{2}}+u_{l-\frac{1}{2}})/2 \quad\text{and}\quad \delta_xu_l=(u_{l+\frac{1}{2}}-u_{l-\frac{1}{2}})$$

are average and central difference operators with respect to x-direction etc. However, the method (5) fails when the solutions are to be determined at $l=1$ and $m=1$ if D and E contains the singularities at $x=0$ and $y=0$. We overcome this difficulty by modifying the method (5) in such a way that the solutions retain the order and accuracy even in the vicinity of the singularities $x=0$ and $y=0$.

Let us use the following approximations:

$$D_{l\pm 1} = D_{00} \pm hD_{10} + \frac{h^2}{2}D_{20} + O(\pm h^3 + h^4), \tag{7a}$$

$$E_{l\pm 1,m} = E_{00} \pm hE_{10} + \frac{h^2}{2}E_{20} + O(\pm h^3 + h^4), \tag{7b}$$

$$E_{l,m\pm 1} = E_{00} \pm hE_{01} + \frac{h^2}{2}E_{02} + O(\pm h^3 + h^4), \tag{7c}$$

$$\bar{G}^j_{l\pm 1,m,n} = G_{000} \pm hG_{100} + \frac{h^2}{2}G_{200} + O(\pm kh \pm h^3 + h^4), \tag{7d}$$

$$\bar{G}^j_{l,m\pm 1,n} = G_{000} \pm hG_{010} + \frac{h^2}{2}G_{020} + O(\pm kh \pm h^3 + h^4), \tag{7e}$$

$$\bar{G}^j_{l,m,n\pm 1} = G_{000} \pm hG_{001} + \frac{h^2}{2}G_{002} + O(\pm kh \pm h^3 + h^4), \tag{7f}$$

where

$$J_{pqr} = \partial^{p+q+r}J(x_l, y_m, z_n, \bar{t}_j)/[(\partial x_l)^p(\partial y_m)^q(\partial z_n)^r], \quad J = G, H, I.$$

Substituting the approximations (7a)–(7f) into (5), neglecting high-order terms and using (3b), we obtain a new difference scheme of $O(k^2 + h^4)$ for solving the equation (1) as

$$\begin{aligned}[L_1][L_2][L_3]u^{j+1}_{l,m,n} = [N_1][N_2][N_3]u^j_{l,m,n} + \frac{\lambda h}{6}\Big[&B_{10}\delta_y^2\mu_x\delta_x + C_{10}\delta_z^2\mu_x\delta_x + C_{01}\delta_z^2\mu_y\delta_y \\ &- hE_{10}\mu_x\delta_x\mu_y\delta_y\Big]u^j_{l,m,n} - \frac{k}{12}\sum G \equiv [R_u],\end{aligned} \tag{8}$$

where

$$\begin{aligned}
&p_{10} = \frac{h^2}{2}(2D_{10} - 2q_1D_{00}), \quad p_{11} = \frac{h^2}{2}(q_3 + 2E_{01} - 2q_2E_{00}), \quad p_{12} = \frac{h^2}{2}q_4, \\
&p_{13} = \frac{h}{4}[12D_{00} + h^2(D_{20} - 2q_1D_{10})], \\
&p_{14} = \frac{h}{4}[12E_{00} + h^2(E_{20} + E_{02} - 2q_1E_{10} - 2q_2E_{01})], \\
&L_1 = 1 + \tfrac{1}{12}(1 - 6\lambda + \lambda p_{10})\delta_x^2 - \tfrac{1}{12}(hq_1 - \lambda p_{13})(2\mu_x\delta_x), \\
&L_2 = 1 + \tfrac{1}{12}(1 - \lambda p_1 + \lambda p_{11})\delta_y^2 - \tfrac{1}{12}(hq_2 - \lambda p_{14})(2\mu_y\delta_y), \\
&L_3 = 1 + \tfrac{1}{12}(1 - \lambda p_2 + \lambda p_{12})\delta_z^2, \\
&N_1 = 1 + \tfrac{1}{12}(1 + 6\lambda - \lambda p_{10})\delta_x^2 - \tfrac{1}{12}(hq_1 + \lambda p_{13})(2\mu_x\delta_x), \\
&N_2 = 1 + \tfrac{1}{12}(1 + \lambda p_1 - \lambda p_{11})\delta_y^2 - \tfrac{1}{12}(hq_2 + \lambda p_{14})(2\mu_y\delta_y), \\
&N_3 = 1 + \tfrac{1}{12}(1 + \lambda p_2 - \lambda p_{12})\delta_z^2, \\
&\sum G = 12G_{000} + h^2(G_{200} + G_{020} + G_{002} - 2q_1G_{100} - 2q_2G_{010}).
\end{aligned} \tag{9}$$

The additional terms are of high orders and do not affect the accuracy of the scheme but enables a factorization of the operators on the left-hand side of (8). Using the Von–Newman method, the amplification factor ξ of (8) can be written as

$$\xi=\xi_1\xi_2\xi_3+\frac{\lambda h}{6d_1d_2d_3}\Big[hE_{10}\sin\beta h\sin\gamma h+2i\{B_{10}\sin\beta h(\cos\gamma h-1)$$

$$+C_{10}\sin\beta h(\cos\psi h-1)+C_{01}\sin\gamma h(\cos\psi h-1)\}\Big], \tag{10}$$

where i is an imaginary quantity and

$$\xi_1=\frac{n_1}{d_1}=\frac{1+\frac{1}{6}(1+6\lambda-\lambda p_{10})(\cos\beta h-1)-\frac{1}{6}i(hq_1+\lambda p_{13})\sin\beta h}{1+\frac{1}{6}(1-6\lambda+\lambda p_{10})(\cos\beta h-1)-\frac{1}{6}i(hq_1-\lambda p_{13})\sin\beta h},$$

$$\xi_2=\frac{n_2}{d_2}=\frac{1+\frac{1}{6}(1+\lambda p_1-\lambda p_{11})(\cos\gamma h-1)-\frac{1}{6}i(hq_2+\lambda p_{14})\sin\gamma h}{1+\frac{1}{6}(1-\lambda p_1+\lambda p_{11})(\cos\gamma h-1)-\frac{1}{6}i(hq_2-\lambda p_{14})\sin\gamma h},$$

$$\xi_3=\frac{n_3}{d_3}=\frac{1+\frac{1}{6}(1+\lambda p_2-\lambda p_{12})(\cos\psi h-1)}{1+\frac{1}{6}(1-\lambda p_2+\lambda p_{12})(\cos\psi h-1)}.$$

For stability we require that $|\xi_j|\leqslant 1$, $j=1,2$ and 3. In order to facilitate the computation we may write the scheme in three-step split form as

$$[L_3]u^{**j+1}_{l,m,n}=[R_u], \tag{11a}$$

$$[L_2]u^{*j+1}_{l,m,n}=u^{**j+1}_{l,m,n}, \tag{11b}$$

$$[L_1]u^{j+1}_{l,m,n}=u^{*j+1}_{l,m,n}. \tag{11c}$$

Let us note the following: The left-hand side of (11) are factorizations into z,y and x differences which allows us to solve (11) by sweeping first in the z-, second in the y- and then in x-direction. It will be seen that these sweeps require only the solution of tridiagonal systems. $u^{*j+1}_{l,m,n}$ and $u^{**j+1}_{l,m,n}$ are intermediate values and the intermediate boundary conditions require for sweeping are obtained from (11b) and (11c).

Consider now, the three space dimensional problem

$$u_{rr}+\frac{1}{r^2}u_{\theta\theta}+\frac{\operatorname{cosec}^2\theta}{r^2}u_{\phi\phi}+\frac{2}{r}u_r+\frac{\cot\theta}{r^2}u_\theta=u_t+G(r,\theta,\phi,t). \tag{12}$$

The above equation represents three space dimensional heat equation in spherical polar coordinates. Replacing the variables x,y,z by r,θ,ϕ, respectively, and setting $B=1/r^2$, $C=\operatorname{cosec}^2\theta/r^2$, $D=-2/r$,

$E = -\cot\theta/r^2$ in (8), we get the ADI method (11), where

$$B_{00} = \frac{1}{(r_l)^2},\quad B_{10} = \frac{-2}{(r_l)^3},\quad B_{20} = \frac{6}{(r_l)^4},\quad D_{00} = \frac{-2}{r_l},$$

$$D_{10} = 2B_{00},\quad D_{20} = 2B_{10},\quad F_{00} = \cot(\theta_m),\quad C_{00} = B_{00}\operatorname{cosec}^2(\theta_m),$$

$$C_{10} = C_{00}D_{00},\quad C_{01} = -2C_{00}F_{00},\quad C_{20} = 6B_{00}C_{00},$$

$$C_{02} = C_{00}[2\operatorname{cosec}^2(\theta_m) + 4(F_{00})^2],\quad E_{00} = -B_{00}F_{00},$$

$$E_{10} = -B_{10}F_{00},\quad E_{01} = C_{00},\quad E_{20} = -B_{20}F_{00},\quad E_{02} = C_{01}$$

and other coefficients associated with (11) are listed in (6) and (9).

Again consider the three space dimensional problem

$$u_{rr} + \frac{1}{r^2}u_{\theta\theta} + u_{zz} + \frac{1}{r}u_r = u_t + G(r,\theta,z,t). \tag{13}$$

Above equation is a three space dimensional heat equation in cylindrical polar coordinates. Replacing the variables x, y by r, θ, respectively and setting $B = 1/r^2$, $C = 1$, $D = -1/r$, $E = 0$ in (8), we get the ADI method (11), where $B_{00} = 1/(r_l)^2$, $B_{10} = -2/(r_l)^3$, $B_{20} = 6/(r_l)^4$, $D_{00} = -1/r_l$, $D_{10} = B_{00}$, $D_{20} = B_{10}$, $C_{00} = 1.0$, $C_{10} = C_{01} = C_{20} = C_{02} = E_{00} = E_{10} = E_{01} = E_{20} = E_{02} = 0$ and the other coefficients associated with (11) are given by (6) and (9). With these coefficients, it is easy to verify from (10) that $|\xi_j| \leqslant 1$, $j = 1, 2$ and 3. Hence, the scheme (11) for (13) is unconditionally stable.

Similarly, for the three space problem

$$u_{rr} + u_{\theta\theta} + u_{zz} + \frac{\alpha}{r}u_r = u_t + G(r,\theta,z,t). \tag{14}$$

We have the ADI method (11) with $B = C = 1.0$, $D = -\alpha/r$, $E = 0$, α being a constant and the variables x, y are replaced by r, z, respectively. The coefficients are given by $B_{00} = C_{00} = 1.0$, $D_{00} = -\alpha/r_l$, $D_{10} = \alpha/(r_l)^2$, $D_{20} = -2\alpha/(r_l)^3$, $B_{10} = B_{20} = C_{10} = C_{01} = C_{20} = C_{02} = E_{00} = E_{10} = E_{01} = E_{20} = E_{02} = 0$ and the other coefficients are listed in (6) and (9). With these coefficients from (10) we can verify that $|\xi_j| \leqslant 1$, $j = 1, 2$ and 3. Hence, the ADI method (11) for (14) is again unconditionally stable.

Note that, the left-hand-side matrices represented by (11) for solving the differential equations (12)–(14) are tridiagonal and are free from the terms $1/(l \pm 1), 1/(m \pm 1)$ and $1/(n \pm 1)$, thus very easily solved for $l, m, n = 1(1)N$ in the region Ω and no fictitious points are required to calculate the intermediate boundary conditions.

Now, let us consider the three space dimensional unsteady Navier–Stokes' model equations for viscous incompressible flow in cylindrical polar coordinates

$$\nu\left[u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}(u_{\theta\theta} - u - 2v_\theta) + u_{zz}\right] = u_t + uu_r + \frac{1}{r}(vu_\theta - v^2) + wu_z + G(r,\theta,z,t), \tag{15a}$$

$$\nu\left[v_{rr} + \frac{1}{r}v_r + \frac{1}{r^2}(v_{\theta\theta} - v + 2u_\theta) + v_{zz}\right] = v_t + uv_r + \frac{1}{r}(vv_\theta + uv) + wv_z + H(r,\theta,z,t), \tag{15b}$$

$$\nu\left[w_{rr} + \frac{1}{r}w_r + \frac{1}{r^2}w_{\theta\theta} + w_{zz}\right] = w_t + uw_r + \frac{1}{r}vw_\theta + ww_z + I(r,\theta,z,t) \tag{15c}$$

subject to appropriate initial and boundary conditions of the type (2), where $Re = \nu^{-1} > 0$ being a constant and represents Reynolds number.

Using same approaches, a difference scheme of $O(k^2 + h^4)$ for (15) may be written as

$$\nu\left[6\delta_r^2 + L_1\delta_\theta^2 + 6\delta_z^2 + L_2(2\delta_\theta^2\mu_r\delta_r) + L_3\delta_r^2\delta_\theta^2 + L_3\delta_\theta^2\delta_z^2 + \delta_r^2\delta_z^2\right]\bar{u}_{l,m,n}^j$$

$$= \frac{h^2}{2}\left[\bar{F}_{l+1,m,n}^{(1)j} + \bar{F}_{l-1,m,n}^{(1)j} + \bar{F}_{l,m+1,n}^{(1)j} + \bar{F}_{l,m-1,n}^{(1)j} + \bar{F}_{l,m,n+1}^{(1)j} + \bar{F}_{l,m,n-1}^{(1)j} + 6\bar{\bar{F}}_{l,m,n}^{(1)j}\right], \tag{16a}$$

$$\nu\left[6\delta_r^2 + L_1\delta_\theta^2 + 6\delta_z^2 + L_2(2\delta_\theta^2\mu_r\delta_r) + L_3\delta_r^2\delta_\theta^2 + L_3\delta_\theta^2\delta_z^2 + \delta_r^2\delta_z^2\right]\bar{v}_{l,m,n}^j$$

$$= \frac{h^2}{2}\left[\bar{F}_{l+1,m,n}^{(2)j} + \bar{F}_{l-1,m,n}^{(2)j} + \bar{F}_{l,m+1,n}^{(2)j} + \bar{F}_{l,m-1,n}^{(2)j} + \bar{F}_{l,m,n+1}^{(2)j} + \bar{F}_{l,m,n-1}^{(2)j} + 6\bar{\bar{F}}_{l,m,n}^{(2)j}\right], \tag{16b}$$

$$\nu\left[6\delta_r^2 + L_1\delta_\theta^2 + 6\delta_z^2 + L_2(2\delta_\theta^2\mu_r\delta_r) + L_3\delta_r^2\delta_\theta^2 + L_3\delta_\theta^2\delta_z^2 + \delta_r^2\delta_z^2\right]\bar{w}_{l,m,n}^j$$

$$= \frac{h^2}{2}\left[\bar{F}_{l+1,m,n}^{(3)j} + \bar{F}_{l-1,m,n}^{(3)j} + \bar{F}_{l,m+1,n}^{(3)j} + \bar{F}_{l,m-1,n}^{(3)j} + \bar{F}_{l,m,n+1}^{(3)j} + \bar{F}_{l,m,n-1}^{(3)j} + 6\bar{\bar{F}}_{l,m,n}^{(3)j}\right], \tag{16c}$$

where we denote

$$L_1 = \frac{6}{l^2h^2} + \frac{3}{l^4h^2}, \quad L_2 = \frac{-1}{l^3h^2}, \quad L_3 = \frac{1}{2}\left(1 + \frac{1}{l^2h^2}\right)$$

and

$$\bar{F}_{l\pm1,m,n}^{(1)j} = \bar{u}_{tl\pm1,m,n}^j + \bar{u}_{l\pm1,m,n}^j\bar{u}_{rl\pm1,m,n}^j + \frac{1}{r_{l\pm1}}\left[\bar{v}_{l\pm1,m,n}^j\bar{u}_{\theta l\pm1,m,n}^j - (\bar{v}_{l\pm1,m,n}^j)^2 - \nu\bar{u}_{rl\pm1,m,n}^j\right]$$
$$+ \bar{w}_{l\pm1,m,n}^j\bar{u}_{zl\pm1,m,n}^j + \frac{\nu}{(r_{l\pm1})^2}(\bar{u}_{l\pm1,m,n}^j + 2\bar{v}_{\theta l\pm1,m,n}^j) + \bar{G}_{l\pm1,m,n}^j,$$

$$\bar{F}_{l\pm1,m,n}^{(2)j} = \bar{v}_{tl\pm1,m,n}^j + \bar{u}_{l\pm1,m,n}^j\bar{v}_{rl\pm1,m,n}^j + \bar{w}_{l\pm1,m,n}^j\bar{v}_{zl\pm1,m,n}^j$$
$$+ \frac{1}{r_{l\pm1}}\left[\bar{v}_{l\pm1,m,n}^j\bar{v}_{\theta l\pm1,m,n}^j + \bar{u}_{l\pm1,m,n}^j\bar{v}_{l\pm1,m,n}^j - \nu\bar{v}_{rl\pm1,m,n}^j\right]$$
$$+ \frac{\nu}{(r_{l\pm1})^2}(\bar{v}_{l\pm1,m,n}^j - 2\bar{u}_{\theta l\pm1,m,n}^j) + \bar{H}_{l\pm1,m,n}^j,$$

$$\bar{F}_{l\pm1,m,n}^{(3)j} = \bar{w}_{tl\pm1,m,n}^j + \bar{u}_{l\pm1,m,n}^j\bar{w}_{rl\pm1,m,n}^j + \bar{w}_{l\pm1,m,n}^j\bar{w}_{zl\pm1,m,n}^j$$
$$+ \frac{1}{r_{l\pm1}}(\bar{v}_{l\pm1,m,n}^j\bar{w}_{\theta l\pm1,m,n}^j - \nu\bar{w}_{rl\pm1,m,n}^j) + \bar{I}_{l\pm1,m,n}^j.$$

Similarly, we can evaluate $\bar{F}^{(1)j}_{l,m\pm1,n}$, $\bar{F}^{(2)j}_{l,m\pm1,n}$, $\bar{F}^{(3)j}_{l,m\pm1,n}$, $\bar{F}^{(1)j}_{l,m,n\pm1}$, $\bar{F}^{(2)j}_{l,m,n\pm1}$, $\bar{F}^{(3)j}_{l,m,n\pm1}$ and

$$\begin{aligned}
\bar{\bar{u}}^j_{rl,m,n} &= \bar{u}^j_{rl,m,n} - \frac{1}{3hl^3}\bar{u}^j_{\theta\theta l,m,n} - \frac{h}{12v}\Bigg[(\bar{F}^{(1)j}_{l+1,m,n} - \bar{F}^{(1)j}_{l-1,m,n}) \\
&\quad - \frac{v}{l^2h^2}(\bar{u}^j_{\theta\theta l+1,m,n} - \bar{u}^j_{\theta\theta l-1,m,n}) - v(\bar{u}^j_{zzl+1,m,n} - \bar{u}^j_{zzl-1,m,n})\Bigg], \\
\bar{\bar{u}}^j_{\theta l,m,n} &= \bar{u}^j_{\theta l,m,n} - \frac{l^2h^3}{12v}\Bigg[(\bar{F}^{(1)j}_{l,m+1,n} - \bar{F}^{(1)j}_{l,m-1,n}) \\
&\quad - v(\bar{u}^j_{rrl,m+1,n} - \bar{u}^j_{rrl,m-1,n}) - v(\bar{u}^j_{zzl,m+1,n} - \bar{u}^j_{zzl,m-1,n})\Bigg], \\
\bar{\bar{u}}^j_{zl,m,n} &= \bar{u}^j_{zl,m,n} - \frac{h}{12v}\Bigg[(\bar{F}^{(1)j}_{l,m,n+1} - \bar{F}^{(1)j}_{l,m,n-1}) \\
&\quad - v(\bar{u}^j_{rrl,m,n+1} - \bar{u}^j_{rrl,m,n-1}) - \frac{v}{l^2h^2}(\bar{u}^j_{\theta\theta l,m,n+1} - \bar{u}^j_{\theta\theta l,m,n-1})\Bigg], \\
\bar{\bar{v}}^j_{rl,m,n} &= \bar{v}^j_{rl,m,n} - \frac{1}{3hl^3}\bar{v}^j_{\theta\theta l,m,n} - \frac{h}{12v}\Bigg[(\bar{F}^{(2)j}_{l+1,m,n} - \bar{F}^{(2)j}_{l-1,m,n}) \\
&\quad - \frac{v}{l^2h^2}(\bar{v}^j_{\theta\theta l+1,m,n} - \bar{v}^j_{\theta\theta l-1,m,n}) - v(\bar{v}^j_{zzl+1,m,n} - \bar{v}^j_{zzl-1,m,n})\Bigg], \\
\bar{\bar{v}}^j_{\theta l,m,n} &= \bar{v}^j_{\theta l,m,n} - \frac{l^2h^3}{12v}\Bigg[(\bar{F}^{(2)j}_{l,m+1,n} - \bar{F}^{(2)j}_{l,m-1,n}) \\
&\quad - v(\bar{v}^j_{rrl,m+1,n} - \bar{v}^j_{rrl,m-1,n}) - v(\bar{v}^j_{zzl,m+1,n} - \bar{v}^j_{zzl,m-1,n})\Bigg], \\
\bar{\bar{v}}^j_{zl,m,n} &= \bar{v}^j_{zl,m,n} - \frac{h}{12v}\Bigg[(\bar{F}^{(2)j}_{l,m,n+1} - \bar{F}^{(2)j}_{l,m,n-1}) \\
&\quad - v(\bar{v}^j_{rrl,m,n+1} - \bar{v}^j_{rrl,m,n-1}) - \frac{v}{l^2h^2}(\bar{v}^j_{\theta\theta l,m,n+1} - \bar{v}^j_{\theta\theta l,m,n-1})\Bigg], \\
\bar{\bar{w}}^j_{rl,m,n} &= \bar{w}^j_{rl,m,n} - \frac{1}{3hl^3}\bar{w}^j_{\theta\theta l,m,n} - \frac{h}{12v}\Bigg[(\bar{F}^{(3)j}_{l+1,m,n} - \bar{F}^{(3)j}_{l-1,m,n}) \\
&\quad - \frac{v}{l^2h^2}(\bar{w}^j_{\theta\theta l+1,m,n} - \bar{w}^j_{\theta\theta l-1,m,n}) - v(\bar{w}^j_{zzl+1,m,n} - \bar{w}^j_{zzl-1,m,n})\Bigg], \\
\bar{\bar{w}}^j_{\theta l,m,n} &= \bar{w}^j_{\theta l,m,n} - \frac{l^2h^3}{12v}\Bigg[(\bar{F}^{(3)j}_{l,m+1,n} - \bar{F}^{(3)j}_{l,m-1,n}) \\
&\quad - v(\bar{w}^j_{rrl,m+1,n} - \bar{w}^j_{rrl,m-1,n}) - v(\bar{w}^j_{zzl,m+1,n} - \bar{w}^j_{zzl,m-1,n})\Bigg],
\end{aligned}$$

$$\bar{\bar{w}}^j_{zl,m,n} = \bar{w}^j_{zl,m,n} - \frac{h}{12v}\Bigg[(\bar{F}^{(3)j}_{l,m,n+1} - \bar{F}^{(3)j}_{l,m,n-1})$$

$$-v(\bar{w}^j_{rrl,m,n+1} - \bar{w}^j_{rrl,m,n-1}) - \frac{v}{l^2h^2}(\bar{w}^j_{00l,m,n+1} - \bar{w}^j_{00l,m,n-1})\Bigg],$$

$$\bar{\bar{F}}^{(1)j}_{l,m,n} = \bar{u}^j_{tl,m,n} + \bar{u}^j_{l,m,n}\bar{\bar{u}}^j_{rl,m,n} + \frac{1}{lh}\left[\bar{v}^j_{l,m,n}\bar{\bar{u}}^j_{0l,m,n} - (\bar{v}^j_{l,m,n})^2 - v\bar{\bar{u}}^j_{rl,m,n}\right]$$

$$+ \bar{w}^j_{l,m,n}\bar{\bar{u}}^j_{zl,m,n} + \frac{v}{l^2h^2}(\bar{u}^j_{l,m,n} + 2\bar{\bar{v}}^j_{0l,m,n}) + \bar{G}^j_{l,m,n},$$

$$\bar{\bar{F}}^{(2)j}_{l,m,n} = \bar{v}^j_{tl,m,n} + \bar{u}^j_{l,m,n}\bar{\bar{v}}^j_{rl,m,n} + \frac{1}{lh}\left[\bar{v}^j_{l,m,n}\bar{\bar{v}}^j_{0l,m,n} + \bar{u}^j_{l,m,n}\bar{v}^j_{l,m,n} - v\bar{\bar{v}}^j_{rl,m,n}\right]$$

$$+ \bar{w}^j_{l,m,n}\bar{\bar{v}}^j_{zl,m,n} + \frac{v}{l^2h^2}(\bar{v}^j_{l,m,n} - 2\bar{\bar{u}}^j_{0l,m,n}) + \bar{H}^j_{l,m,n},$$

$$\bar{\bar{F}}^{(3)j}_{l,m,n} = \bar{w}^j_{tl,m,n} + \bar{u}^j_{l,m,n}\bar{\bar{w}}^j_{rl,m,n} + \frac{1}{lh}\left[\bar{v}^j_{l,m,n}\bar{\bar{w}}^j_{0l,m,n} - v\bar{\bar{w}}^j_{rl,m,n}\right] + \bar{w}^j_{l,m,n}\bar{\bar{w}}^j_{zl,m,n} + \bar{I}^j_{l,m,n}$$

and in which we use the following approximations:

$$\frac{1}{r_{l\pm1}} = \frac{1}{r_l} \mp \frac{h}{(r_l)^2} + \frac{h^2}{(r_l)^3} + \mathrm{O}(\pm h^3 + h^4), \tag{17a}$$

$$\frac{1}{(r_{l\pm1})^2} = \frac{1}{(r_l)^2} \mp \frac{2h}{(r_l)^3} + \frac{3h^2}{(r_l)^4} + \mathrm{O}(\pm h^3 + h^4), \tag{17b}$$

$$\bar{G}^j_{l\pm1,m,n} = G_{000} \pm hG_{100} + \frac{h^2}{2}G_{200} + \mathrm{O}(\pm kh \pm h^3 + h^4), \tag{17c}$$

$$\bar{G}^j_{l,m\pm1,n} = G_{000} \pm hG_{010} + \frac{h^2}{2}G_{020} + \mathrm{O}(\pm kh \pm h^3 + h^4), \tag{17d}$$

$$\bar{G}^j_{l,m,n\pm1} = G_{000} \pm hG_{001} + \frac{h^2}{2}G_{002} + \mathrm{O}(\pm kh \pm h^3 + h^4), \tag{17e}$$

$$\bar{H}^j_{l\pm l,m,n} = H_{000} \pm hH_{100} + \frac{h^2}{2}H_{200} + \mathrm{O}(\pm kh \pm h^3 + h^4), \tag{17f}$$

$$\bar{H}^j_{l,m\pm1,n} = H_{000} \pm hH_{010} + \frac{h^2}{2}H_{020} + \mathrm{O}(\pm kh \pm h^3 + h^4), \tag{17g}$$

$$\bar{H}^j_{l,m,n\pm1} = H_{000} \pm hH_{001} + \frac{h^2}{2}H_{002} + \mathrm{O}(\pm kh \pm h^3 + h^4), \tag{17h}$$

$$\bar{I}^j_{l\pm1,m,n} = I_{000} \pm hI_{100} + \frac{h^2}{2}I_{200} + \mathrm{O}(\pm kh \pm h^3 + h^4), \tag{17i}$$

$$\bar{I}^j_{l,m\pm1,n} = I_{000} \pm hI_{010} + \frac{h^2}{2}I_{020} + \mathrm{O}(\pm kh \pm h^3 + h^4), \tag{17j}$$

$$\bar{I}^j_{l,m,n\pm1} = I_{000} \pm hI_{001} + \frac{h^2}{2}I_{002} + \mathrm{O}(\pm kh \pm h^3 + h^4). \tag{17k}$$

In a similar manner, we can write a two-level implicit scheme of $O(k^2+h^4)$ using 19-spatial grid points for three space dimensional unsteady Navier–Stokes' equations in spherical polar coordinates. Note that, the scheme (16) along with the approximations (17) is of $O(k^2+h^4)$ and free from the terms $1/(l\pm 1), 1/(m\pm 1)$ and $1/(n\pm 1)$, hence very easily solved for $l,m,n=1(1)N$ in the region Ω.

If $r=0$ is a part of the boundary and the solution at $r=0$ is also to be determined then we need a difference equation valid at $r=0$. We illustrated the procedure for the Eq. (14). Let at $r=0$, we have $u_r=0$, then Eq. (14) becomes

$$(1+\alpha)u_{rr}+u_{\theta\theta}+u_{zz}=u_t+G(0,\theta,z,t). \tag{18}$$

A suitable $O(k^2+h^4)$ approximation is

$$[L_4][L_5][L_6]u^{j+1}_{0,m,n}=[N_4][N_5][N_6]u^{j}_{0,m,n} - \frac{k}{12}\left[12G_{0,m,n}+h^2(G_{rr0,m,n}+G_{\theta\theta 0,m,n}+G_{zz0,m,n})\right]\equiv[S_u], \tag{19}$$

where

$$L_4=1+\tfrac{1}{12}[1-6\lambda(1+\alpha)]\delta_r^2,\qquad L_5=1+\tfrac{1}{12}(1-6\lambda)\delta_\theta^2,$$
$$L_6=1+\tfrac{1}{12}(1-6\lambda)\delta_z^2,\qquad N_4=1+\tfrac{1}{12}[1+6\lambda(1+\alpha)]\delta_r^2,$$
$$N_5=1+\tfrac{1}{12}(1+6\lambda)\delta_\theta^2,\qquad N_6=1+\tfrac{1}{12}(1+6\lambda)\delta_z^2.$$

A split form is

$$[L_6]u^{**j+1}_{0,m,n}=[S_u], \tag{20a}$$

$$[L_5]u^{*j+1}_{0,m,n}=u^{**j+1}_{0,m,n}, \tag{20b}$$

$$[L_4]u^{j+1}_{0,m,n}=u^{*j+1}_{0,m,n}, \tag{20c}$$

where in (19) we are to use the condition $u_r=0$ at $r=0$. This implies $u^{**j+1}_{-1,m,n}=u^{**j+1}_{1,m,n}$ and $u^{*j+1}_{-1,m,n}=u^{*j+1}_{1,m,n}$. Note that, Eqs. (11b) and (20b) are same and (11c) and (20c) are same valid along lines parallel to θ and r-axis, respectively. The intermediate boundary conditions are obtained from (20b) and (20c).

In all above ADI methods, the integration is first carried along lines parallel to the ϕ-axis or z-axis, then along θ-axis and then along r-axis. These methods produce tridiagonal systems for solution along lines parallel to the axis. They are all two-level formulas so that no extra starting values are required and these methods are unconditionally stable so that large step length along the t-direction may be used. These methods are also of $O(k^2+h^4)$. For a fixed $\lambda=k/h^2$, all the above methods behave like fourth-order methods.

3. Computational implementation

In this section, we have compared the numerical solution of (12)–(15) with exact solutions on the specified region Ω. The initial, Dirichlet boundary conditions and right-hand-side functions may

Table 1
Example 1. The RMS errors

h ↓	Example 1(a)	Example 1(b)	Example 1(c)	
			$\alpha=1$	$\alpha=2$
$\frac{1}{4}$	0.1013(−04)	0.7905(−03)	0.5381(−03)	0.7385(−03)
$\frac{1}{8}$	0.6553(−06)	0.5875(−04)	0.2932(−04)	0.4079(−04)
$\frac{1}{16}$	0.4136(−07)	0.3303(−05)	0.1699(−05)	0.2363(−05)
$\frac{1}{32}$	0.2987(−08)	0.2560(−06)	0.1029(−06)	0.1422(−06)

Table 2
Example 2. The RMS errors

h ↓	Re→	10	20	25	40	50
$\frac{1}{4}$	u	0.2488(−03)	0.3954(−03)	0.4563(−03)	0.5478(−03)	0.5764(−03)
	v	0.6603(−04)	0.7082(−04)	0.7493(−04)	0.1273(−03)	0.1827(−03)
	w	0.9403(−04)	0.1268(−03)	0.1424(−03)	0.2145(−03)	0.2727(−03)
$\frac{1}{8}$	u	0.1083(−04)	0.2129(−04)	0.3232(−04)	0.4623(−04)	0.4986(−04)
	v	0.5129(−05)	0.5832(−05)	0.6318(−05)	0.9020(−05)	0.1012(−04)
	w	0.6618(−05)	0.8619(−05)	0.8921(−05)	0.1156(−04)	0.1261(−04)

be obtained using the exact solutions. The linear equations have been solved using the ADI method whereas the system of nonlinear equations have been solved using the Newton–Raphson method. All computations were carried out using double precision arithmetic at the computer service centre, University of Delhi.

Example 1. The problems are to solve (12) and (13) using the scheme (11) with the exact solutions (a) $u=e^{-4t}r^2\phi\cos\theta$ and (b) $u=e^{-2t}r^2\cos(\pi\theta)\cos(\pi z)$, respectively. (c) The problem is to solve (14) using the scheme (11) by the help of the scheme (19) with the exact solution $u=e^{-t}\cosh r\cosh\theta\cosh z$, where $u_r=0$ at $r=0$ and all other conditions are provided from the exact solutions.

The root-mean-square (RMS) errors are tabulated in Table 1 at $t=1.0$ for a fixed $\lambda=1.6$.

Example 2. The problem is to solve (15) using the difference scheme (16) along with approximations (17) with exact solutions $u=e^{-t}r^3\sin\theta\cos z$, $v=e^{-t}r^3\cos\theta\cos z$ and $w=-3e^{-t}r^2\sin\theta\sin z$. The RMS errors are tabulated in Table 2 at $t=0.5$ for various values of Re for a fixed $\lambda=1.6$.

4. Concluding remarks

In this article, we have proposed two-level implicit difference scheme of $O(k^2+h^4)$ using 19-spatial grid points for solving the three space dimensional heat conduction and unsteady Navier–Stokes' equations for viscous, incompressible flow in polar coordinates. We have also been shown that the proposed ADI schemes are unconditionally stable. These methods give accurate results everywhere including the region in the vicinity of $r=0$. A special treatment is required if $r=0$ is a part of the boundary and the solution is to be determined at $r=0$. This is illustrated for Eq. (14) in the r–θ–z space. The solutions remain accurate at all points on $r=0$ and the order is preserved. From the numerical results, we found that the proposed difference methods are of $O(h^4)$ for a fixed λ and very near to accurate one. During computation we found that the linear scheme takes reasonably less computer time, whereas the system of nonlinear schemes take large computer time to achieve fourth-order accuracy of the solution. In the case of nonlinear equations for $h=\frac{1}{16}$ and $\frac{1}{32}$, it is required to solve simultaneously three system of large band width (19-diagonal) matrices of order $15^3\times 15^3$ and $31^3\times 31^3$ at each time level. So in order to avoid a large computer time, we have restricted to $h=\frac{1}{4}$ and $\frac{1}{8}$ for the Eq. (15) only. Further, during computation we found that the proposed system of non-linear difference schemes (16) for Navier–Stokes equations (15) are unstable for $Re>50$, whereas for steady-state case (where $u^{j+1}=u^j=u$), the schemes are even stable for $Re=10^2, 10^4$, etc., i.e. for high Reynolds numbers (see [9]).

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